Lecture 15 Summary Thurs 21/08/14

Vocabulary

(No new vocabulary defined).

Examples

COMPLETIONS

(1) $\mathbb{R}$ is the completion of $\mathbb{Q}$.

(2) $\mathbb{R}[X]$ = polynomials in one variable X

$= \{a_0 + a_1X + a_2X^2 + \dots + a_\ell X^\ell \mid \ell \in \mathbb{Z}_{\geq 0} \text{ and } a_i \in \mathbb{R}\}$

$= \{\sum_{i\in\mathbb{Z}_{\geq 0}} a_i X^i \mid a_i \in \mathbb{R} \text{ and all but a finite number of } a_i \text{ are } 0\}$

$\mathbb{R}[[X]] = \{\sum_{i\in\mathbb{Z}_{\geq 0}} a_iX^i \mid a_i \in \mathbb{R}\}$ "ring of formal power series"

is the completion of $\mathbb{R}[X]$ with $d(f,g) = \|f-g\|$

where $\|a_0 + a_1X + \dots\| = e^{-val_x(a_0+a_1X+\dots)}$

and $val_x(a_0 + a_1X + \dots) = \min\{k \in \mathbb{Z}_{\geq 0} \mid a_k \neq 0\}$ "x-adic valuation"

(3) Let $p \in \mathbb{Z}_{>0}$ be a prime. If $n \in \mathbb{Z}$ then $n = a_0 + a_1p + a_2p^2 + \dots$

e.g. if $p=3$ then $56 = 2\cdot 3^3 + 2\cdot 3^0 = 2\cdot 3^0 + 2\cdot 3^3$

Introduce $\mathbb{Z}_p = \{\sum_{i\in\mathbb{Z}_{\geq 0}} a_ip^i \mid a_i \in \{0,1,\dots,p-1\}\}$ "p-adic integers"

Using p-adic metric on $\mathbb{Z} = \{\sum_{i\in\mathbb{Z}_{\geq 0}} a_ip^i \mid a_i \in \{0,1,\dots,p-1\}$ and all but a finite number of a_i are $0\}$,

$\mathbb{Z}_p$ is the completion of $\mathbb{Z}$.

(4) $\mathbb{Q} = \{\frac{a}{b} \mid a,b \in \mathbb{Z} \text{ and } b \neq 0\}$ with $\frac{a}{b} = \frac{c}{d}$ if $ad = bc$

$\mathbb{Q}_p = \{\frac{a}{b} \mid a,b \in \mathbb{Z}_p \text{ and } b \neq 0\}$ with $\frac{a}{b} = \frac{c}{d}$ if $ad = bc$ "p-adic numbers"

In the p-adic metric, $\mathbb{Q} \hookrightarrow \mathbb{Q}_p$ and $\overline{\mathbb{Q}} = \mathbb{Q}_p$.

Homework

- Show that $\mathbb{R}[[X]]$ is a completion of $\mathbb{R}[X]$

  ie. $\mathbb{R}[X] \hookrightarrow \mathbb{R}[[X]]$ and $\overline{\mathbb{R}[X]} = \mathbb{R}[[X]]$.